An easier and more awesome proof of the FTC

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Wednesday, March 23rd, 2011

The book gives a pretty complicated and non-standard proof of the FTC. Here's a more straightforward version which I hope is easier to understand!

1 FTC - Part II

I will start with FTC - Part II, because then FTC - Part I will be easier to prove!

Theorem (FTC - Part II)**.** *If f is integrable on $[a,b]$, then:*

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f

The proof of this is based on two easy facts:

Fact 1.

$$\frac{F(x_i) - F(x_{i-1})}{\frac{b-a}{n}} = f(c_i)$$

for some c_i in (x_{i-1}, x_i)

Proof. This is just the MVT on (x_{i-1}, x_i). Namely, there is a c_i in (x_{i-1}, x_i) such that:

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i)$$

However, $\boxed{F'(c_i) = f(c_i)}$ since F is an antiderivative of f.

Moreover, $\boxed{x_i - x_{i-1} = \Delta x = \frac{b-a}{n}}$ (remember that $x_i - x_{i-1}$ is just the width of the i-th rectangle). So putting those two facts together, we get:

$$\frac{F(x_i) - F(x_{i-1})}{\frac{b-a}{n}} = f(c_i)$$

□

Fact 2.

$$\sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = F(b) - F(a)$$

Proof. Convince yourself that this is true by writing out that sum for a couple of values of n. Let's do this for $n = 4$:

$$\begin{aligned}\sum_{i=1}^{4} F(x_i) - F(x_{i-1}) =& \cancel{F(x_1)} - F(x_0) + \cancel{F(x_2)} - \cancel{F(x_1)} + \cancel{F(x_3)} - \cancel{F(x_2)} + F(x_4) - \cancel{F(x_3)} \\ =& F(x_4) - F(x_0) \\ =& F(b) - F(a) \qquad \text{(here } x_4 = b \text{ and } x_0 = a\text{)}\end{aligned}$$

This kind of sum where 'almost everything cancels out' is called a 'telescoping sum'. You will encounter many more telescoping sums in Math 1B.

□

Now that we know those two facts, the proof of the FTC is very easy:

Proof. By definition:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*)$$

Since f is integrable on $[a, b]$, we can choose **any** x_i^*, as long as it is in (x_{i-1}, x_i). Now the trick is: Choose $\boxed{x_i^* = c_i}$, wher c_i is given by **Fact 1**. Then we get:

$$\begin{aligned}\int_a^b f(x)dx =& \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(c_i) \\ =& \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \frac{F(x_i) - F(x_{i-1})}{\frac{b-a}{n}} \qquad \text{(by Fact 1)} \\ =& \lim_{n\to\infty} \left(\cancel{\frac{b-a}{n}}\right) \frac{1}{\cancel{\frac{b-a}{n}}} \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) \qquad \left(\frac{1}{\frac{b-a}{n}} \text{ doesn't depend on } i\text{, so take it out of the sum}\right) \\ =& \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) \\ =& \lim_{n\to\infty} F(b) - F(a) \qquad \text{(by Fact 2)} \\ =& F(b) - F(a) \qquad (F(b) - F(a) \text{ is just a constant!})\end{aligned}$$

□

2 FTC - Part I

Now that we know FTC - Part II, proving FTC - Part I is a breeze!

Theorem (FTC - Part I). *If f is integrable on $[a, b]$, then:*

$$\left(\int_a^x f(t)dt\right)' = f(x)$$

Proof. Let F be an antiderivative of f. Then:

$$\left(\int_a^x f(t)dt\right)' = (F(x) - F(a))' = (F(x))' - (F(a))' = f(x) - 0 = f(x)$$

Where we used FTC - Part II, the fact that $(F(x))' = f(x)$ and $(F(a))' = 0$ (because $F(a)$ is just a constant!) □